

# The CNOT Quantum Logic Gate Using q-Deformed Oscillators

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## Abstract

It is shown that the two qubit CNOT (controlled NOT) gate can also be realised using q-deformed angular momentum states constructed via the Jordan-Schwinger mechanism. Thus all the three gates necessary for universality i.e. Hadamard, Phase Shift and the two qubit CNOT gate are realisable with q-deformed oscillators.

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## 1. Introduction

Recently<sup>1</sup> it has been shown that the single qubit quantum logic gates, *viz.* ,the Hadamard and Phase Shift gates can also be realised with two q-deformed oscillators where  $q$  is the deformation parameter of a quantum Lie algebra<sup>2</sup>. q-Deformed oscillators here mean that the Lie algebra satisfied by creation ( $a^\dagger$ ) and annihilation ( $a$ ) operators of a bosonic oscillator, *viz.*  $aa^\dagger - a^\dagger a = 1$  is modified into  $a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}$  where  $N$  is the number operator and  $q = e^s$  ;  $0 < s < 1$ . Using such deformed oscillators an alternative formalism for quantum computation can be set up<sup>1</sup>. The advantage of this over the conventional formalism (which is obtained for  $q \rightarrow 1$ ) is the presence of an arbitrary function which may be exploited for experimental purposes.

However, the formalism will be more meaningful if the realisation with q-deformed qubits is possible for all the gates required for *universality*. A set of gates is said to be *universal for quantum computation* if any unitary operation may be approximated to arbitrary accuracy by a quantum circuit involving those gates. In the case of standard quantum computation the Hadamard, Phase Shift and the CNOT (controlled NOT) gates constitute such a set<sup>3</sup>. In this paper I show that the 2-qubit controlled-NOT gate can also be realised with q-deformed qubits. Thus all the three gates, i.e. Hadamard, Phase Shift and CNOT gates can now be obtained with q-deformed qubits.

The motivation for considering q-deformed oscillators in quantum computation comes from the fact that deformed oscillators have been successfully used as a tool to understand deviations from an ideal theoretical or experimental scenario<sup>4</sup> for past many years. Bonatsos and Daskoloyamis<sup>4</sup> were among the firsts to show that the vibration spectra of diatomic molecules

gave better fits using deformed oscillators. Parisi<sup>4</sup> studied a d-dimensional array of Josephson junctions in a magnetic field and computed the thermodynamic properties in the high temperature region for  $d \rightarrow \infty$ . Evaluation of the high temperature expansion coefficients were done by mapping onto the computation of some matrix elements for the q-deformed harmonic oscillator. Raychev<sup>4</sup> *et al* calculated the deviations from the nuclear shell model using the q-deformed three dimensional harmonic oscillator. Bonatsos, Lewis, Raychev and Terziev<sup>4</sup> demonstrated that the three dimensional q-deformed harmonic oscillator correctly predicts the first supershell closure in alkali clusters without introducing additional parameters. McEWan and Freer<sup>4</sup> showed that the nuclear orbitals of certain nuclei were commensurate with the energy level scheme of the deformed harmonic oscillator and the Nilsson model. For these reasons it is meaningful to study whether deformed oscillators can be used in the formalism of quantum computation.

A brief review of relevant facts is given in section 2. In Section 3 the NOT gate is realised with q-deformed oscillators. Section 4 gives the realisation of the two qubit CNOT gate in terms of q-deformed oscillators. In Section 5 the states are discussed and Section 6 is the conclusion where a brief elaboration of the possibility of quantum error correction using deformed oscillators is given.

## 2. Brief Review

Quantum logic gates are basically unitary operators<sup>5-9</sup>. Three gates, the single qubit Hadamard and Phase Shift gates and the 2-qubit CNOT gate, are sufficient to construct any unitary operation on a single qubit<sup>3</sup>. This is the *universality* referred to above. These gates are constructed using the "spin

up" and "spin down" states of  $SU(2)$  angular momentum i.e., the two possible states of a qubit are usually represented by "spin up" and "spin down" states. This is the Jordan-Schwinger construction using two independent harmonic oscillators. A similar construction of q-deformed angular momentum states can be done using two q-deformed oscillators<sup>10</sup>. In Ref.1 it was shown that the Hadamard and Phase Shift gates can also be realised with q-deformed qubits. To achieve this ,the technique of harmonic oscillator realisation <sup>11,12</sup> of q-oscillators was used. This allows one to set up an alternate quantum computation formalism.

q-Oscillators are described by deformed creation and annihilation operators,  $a_q^\dagger, a_q$  ,respectively. For ordinary oscillators these are  $a^\dagger$  and  $a$ .  $q = e^s$ ,  $0 \leq s \leq 1$  and the deformed oscillators satisfy the following relations :

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N} \quad ; \quad N^\dagger = N \quad (1a)$$

$$[N, a_q] = -a_q \quad ; \quad [N, a_q^\dagger] = a_q^\dagger \quad ; \quad a_q^\dagger a_q = [N] \quad ; \quad a_q a_q^\dagger = [N+1] \quad (1b)$$

$$a_q f(N) = f(N+1) a_q \quad ; \quad a_q^\dagger f(N) = f(N-1) a_q^\dagger \quad (1c)$$

where the q-number  $[x] = (q^x - q^{-x})/(q - q^{-1})$  becomes the ordinary number  $x$  when  $q \rightarrow 1$  (i.e.  $s \rightarrow 0$ ).  $N$  is the number operator for the q-deformed oscillators and  $f(N)$  is any function of  $N$ . The eigenvalue  $n$  of the number operator  $N$  denotes the number of bosonic particles. We confine to real  $q$ .  $a_q, a_q^\dagger$  and  $a, a^\dagger$  are related as <sup>11</sup>

$$a_q = a \sqrt{\frac{q^{\hat{N}} \psi_1 - q^{-\hat{N}} \psi_2}{\hat{N}(q - q^{-1})}} \quad ; \quad a_q^\dagger = \sqrt{\frac{q^{\hat{N}} \psi_1 - q^{-\hat{N}} \psi_2}{\hat{N}(q - q^{-1})}} a^\dagger \quad (2a)$$

$$N = \hat{N} - (1/s) \ln \psi_2 \quad (2b)$$

$\hat{N}$  is the number operator for usual oscillators with eigenvalue  $\hat{n}$ ; and  $\psi_1, \psi_2$  are arbitrary functions of  $q$  only with  $\psi_{1,2}(q) = 1$  for  $q = 1$ .

If all these arbitrary functions are unity, then  $N = \hat{N}$ , i.e. the number operator of the deformed oscillator becomes identical to the number operator of the standard harmonic oscillator. But oscillator states are usually expressed in occupation number basis. So if the number operators are identical, there is no way of differentiating between a deformed oscillator state and a standard oscillator state. So nothing is gained and we are still in the realm of standard quantum computation. But equations (2a, b) are general if the  $\psi_i(q), i = 1, 2$  are *not all equal to unity*. Let  $\psi_1 = \psi_2 = \psi(q)$ . Now  $N = \hat{N} - (1/s) \ln \psi(q)$  (equation (2b)). This will be reflected in the Jordan-Schwinger construction of angular momentum states and the states in the two cases will be distinguishable through the function  $\psi(q)$ . Further details are in Ref.[1].

We now express a single qubit state in terms of two harmonic oscillator states using the Jordan-Schwinger construction.

(a) States are defined by the total angular momentum  $j$  and  $z$ -component of angular momentum  $j_z$  i.e.  $m$ . A particular  $(j, m)$  state is created by acting the creation operators on the vacuum or ground state which is a direct product state of the individual oscillator ground states :

$$|jm\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{[(j+m)!(j-m)!]^{1/2}} |\phi\rangle \quad (3)$$

$|\phi\rangle \equiv |\tilde{0}\rangle = |\tilde{0}\rangle_1 |\tilde{0}\rangle_2$  is the ground state ( $j = 0, m = 0$ ), while  $|\tilde{0}\rangle_i, i = 1, 2$  are the oscillator ground states.  $j = (n_1 + n_2)/2$  ;  $m = (n_1 - n_2)/2$  where  $n_1, n_2$  are the eigenvalues of the number operators of the two oscillators.

(b) A qubit can be either "up" or "down" i.e. there are two possible configurations. So the oscillator number operators can take the following sets of values only :  $(n_1 = 1, n_2 = 0, \text{"up" state})$  and  $(n_1 = 0, n_2 = 1, \text{"down" state})$ . Hence  $(n_1 + n_2)/2 = 1/2$ . As  $j = 1/2$  for both qubit states, suppress  $j$  for simplicity of notation :

$$|m\rangle = \frac{(a_1^\dagger)^{1/2+m}(a_2^\dagger)^{1/2-m}}{[(1/2+m)!(1/2-m)!]^{1/2}}|\phi\rangle ; \quad |-m\rangle = \frac{(a_1^\dagger)^{1/2-m}(a_2^\dagger)^{1/2+m}}{[(1/2+m)!(1/2-m)!]^{1/2}}|\phi\rangle \quad (4a)$$

Equivalently, in terms of  $n_1, n_2$  these are

$$|n_1 - 1/2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{1-n_1}}{[(n_1)!(1-n_1)!]^{1/2}}|\tilde{0}\rangle ; \quad |-(n_1 - 1/2)\rangle = \frac{(a_1^\dagger)^{1-n_1}(a_2^\dagger)^{n_1}}{[(n_1)!(1-n_1)!]^{1/2}}|\tilde{0}\rangle \quad (4b)$$

(c) The basis states are  $|1\rangle \equiv |up\rangle$  state and  $|0\rangle \equiv |down\rangle$  state )

$$|1\rangle \equiv |1/2, 1/2\rangle \equiv |1/2\rangle = a_1^\dagger|\tilde{0}\rangle = a_1^\dagger|\tilde{0}\rangle_1|\tilde{0}\rangle_2 = |\tilde{1}\rangle_1|\tilde{0}\rangle_2$$

$$|0\rangle \equiv |1/2, -1/2\rangle \equiv |-1/2\rangle = a_2^\dagger|\tilde{0}\rangle = a_2^\dagger|\tilde{0}\rangle_1|\tilde{0}\rangle_2 = |\tilde{0}\rangle_1|\tilde{1}\rangle_2$$

The *physical meaning* of the notation is as follows. The  $|1\rangle$  (spin "up") state is constructed out of two oscillator states where the first oscillator state has occupation number 1 while the other has occupation number 0. The  $|0\rangle$  ( spin "down") state corresponds to the first oscillator having occupation number 0 and the second oscillator having occupation number 1. So any qubit state  $|x\rangle$  is :

$$|x\rangle = (a_1^\dagger)^x(a_2^\dagger)^{1-x}|\tilde{0}\rangle \quad (5)$$

$|0\rangle$  represents one of the two possible qubit states while  $|\tilde{0}\rangle$  represents oscillator ground state i.e. occupation number 0;  $|\tilde{1}\rangle$  represents an oscillator state with occupation number 1 etc.

Now consider the Hadamard transformation. For a standard qubit, the Hadamard gate acts as follows. When one of the basis states is given as an input, the output is a superposition of the two basis states, i.e.  $|0\rangle \rightarrow |0\rangle + |1\rangle$  and  $|1\rangle \rightarrow |0\rangle - |1\rangle$ . So the Hadamard transformation on a single qubit state ( $x = 0, 1$ ) can be represented as (modulo  $1/\sqrt{2}$ )

$$|x\rangle \longrightarrow (-1)^x |x\rangle + |1-x\rangle \quad (6)$$

i.e.

$$|n_1 - 1/2\rangle \longrightarrow (-1)^{n_1} |n_1 - 1/2\rangle + |1/2 - n_1\rangle \quad (7)$$

Following the discussion preceding equation (3), the general q-deformed state is  $|jm\rangle_q \equiv \frac{(a_{1q}^\dagger)^{n_1} (a_{2q}^\dagger)^{n_2}}{([n_1]! [n_2]!)^{1/2}} |\phi\rangle_q$ ;  $|j - m\rangle_q \equiv \frac{(a_{1q}^\dagger)^{n_2} (a_{2q}^\dagger)^{n_1}}{([n_1]! [n_2]!)^{1/2}} |\phi\rangle_q$  where  $|\phi\rangle_q \equiv |\tilde{0}\rangle_q = |\tilde{0}\rangle_{1q} |\tilde{0}\rangle_{2q}$  is the ground state corresponding to two non-interacting q-deformed oscillators<sup>10</sup>. In our notation a qubit state has either (a)  $n_1 = 0, n_2 = 1$  or (b)  $n_1 = 1, n_2 = 0$ . Hence

$$|n_1 - 1/2\rangle_q \equiv \frac{(a_{1q}^\dagger)^{n_1} (a_{2q}^\dagger)^{1-n_1}}{([n_1]! [1-n_1]!)^{1/2}} |\tilde{0}\rangle_q ; \quad |-(n_1 - 1/2)\rangle_q \equiv \frac{(a_{1q}^\dagger)^{1-n_1} (a_{2q}^\dagger)^{n_1}}{([n_1]! [1-n_1]!)^{1/2}} |\tilde{0}\rangle_q \quad (8)$$

So the Hadamard transformation for q-deformed state is

$$|n_1 - 1/2\rangle_q \longrightarrow (-1)^{n_1} |n_1 - 1/2\rangle_q + |1/2 - n_1\rangle_q \quad (9)$$

This simplifies to:

$$\begin{aligned} & [F_1(\hat{N}_1, q) a_1^\dagger]^{n_1} [F_2(\hat{N}_2, q) a_2^\dagger]^{1-n_1} |\phi\rangle_q \longrightarrow \\ & (-1)^{n_1} [F_1(\hat{N}_1, q) a_1^\dagger]^{n_1} [F_2(\hat{N}_2, q) a_2^\dagger]^{1-n_1} |\phi\rangle_q \\ & + [F_1(\hat{N}_1, q) a_1^\dagger]^{1-n_1} [F_2(\hat{N}_2, q) a_2^\dagger]^{n_1} |\phi\rangle_q \end{aligned} \quad (10)$$

where

$$F_1(\hat{N}_1, q) = \sqrt{\frac{q^{\hat{N}_1}\psi_1 - q^{-\hat{N}_1}\psi_2}{\hat{N}_1(q - q^{-1})}}, \quad F_2(\hat{N}_2, q) = \sqrt{\frac{q^{\hat{N}_2}\psi_3 - q^{-\hat{N}_2}\psi_4}{\hat{N}_2(q - q^{-1})}} \quad (11)$$

$n_1, n_2$  is always 0 or 1 so as to correspond to the qubit. It is simple to check that the q-number  $[0]$  is equal to the ordinary number 0 and similarly the q-number  $[1]$  equals ordinary number 1. Hence the q-numbers  $[n_1], [n_2]$  are always the usual numbers  $n_1, n_2$ . Same restrictions also apply to usual (i.e. undeformed) oscillators. So we restrict the hatted number operators,  $\hat{N}_1$  and  $\hat{N}_2$ , by  $\hat{N}_1 + \hat{N}_2 = I$  where  $I$  is the identity operator.

The Phase Shift transformation of qubit states is :  $|x\rangle \longrightarrow e^{ix\theta}|x\rangle$  i.e  $|0\rangle \rightarrow |0\rangle$  and  $|1\rangle \rightarrow e^{i\theta}|1\rangle$ . which in our notation is  $|n - \frac{1}{2}\rangle \longrightarrow e^{in\theta}|n - \frac{1}{2}\rangle$  where  $\theta$  is the phase shift. Now one can proceed as described in the previous sections. Details are in Ref.[1]. There it was shown that both the Hadamard and Phase Shift transformations can be realised with q-deformed qubits. Below it is shown that the same is possible for both the NOT gate and the CNOT gate.

### 3.The NOT gate

The NOT gate essentially flips a qubit, i.e.  $|0\rangle \rightarrow |1\rangle$  and  $|1\rangle \rightarrow |0\rangle$ . It acts on a qubit as :  $|x\rangle \rightarrow |1 - x\rangle$  where  $x = 0, 1$ . In our notation this is  $|n_1 - \frac{1}{2}\rangle \rightarrow |\frac{1}{2} - n_1\rangle$ . For q-deformed states this means  $|n_1 - \frac{1}{2}\rangle_q \rightarrow |\frac{1}{2} - n_1\rangle_q$ . In terms of q-deformed oscillator states this becomes

$$\frac{(a_{1q}^\dagger)^{n_1}(a_{2q}^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_q \rightarrow \frac{(a_{1q}^\dagger)^{n_2}(a_{2q}^\dagger)^{n_1}}{([n_2]![n_1]!)^{1/2}}|\phi\rangle_q \quad (12)$$

i.e. the exponents of the two creation operators are interchanged. Rewritten



in terms of the functions  $F$  this looks like

$$[F(\hat{N})a_1^\dagger]^n[F(1 - \hat{N})a_2^\dagger]^{1-n}|\phi\rangle_q \rightarrow [F(\hat{N})a_1^\dagger]^{1-n}[F(1 - \hat{N})a_2^\dagger]^n|\phi\rangle_q$$

where one has used the fact that  $n_1 + n_2 = 1$  and followed the arguments *after equation (13b) of Ref.1*, relabelled  $n_1$  as  $n$  etc. Using (1c) one gets

$$\begin{aligned} & [F(\hat{N})]^n[F(1 + n - \hat{N})]^{1-n}(a_1^\dagger)^n(a_2^\dagger)^{1-n}|\phi\rangle_q \\ & \rightarrow [F(\hat{N})]^{1-n}[F(2 - n - \hat{N})]^n(a_1^\dagger)^{1-n}(a_2^\dagger)^n|\phi\rangle_q \end{aligned} \quad (13)$$

With respect to the states  $|\phi\rangle_q$ , the above expression would be indistinguishable from the usual "NOT" transformation if

$$[F(\hat{N})]^n[F(1 + n - \hat{N})]^{1-n} = [F(\hat{N})]^{1-n}[F(2 - n - \hat{N})]^n$$

which simplifies to

$$F(\hat{N}) = F(1 - \hat{N}) \quad (14)$$

for both  $n = 0$  and  $n = 1$ .

Written in terms of its eigenvalues means

$$\frac{\psi_1(q)}{\psi_2(q)} = \frac{(q^{-\hat{n}} - \hat{n}q^{-\hat{n}} - \hat{n}q^{\hat{n}-1})}{(q^{\hat{n}} - \hat{n}q^{\hat{n}} - \hat{n}q^{1-\hat{n}})} \quad (15)$$

This has the solution  $\psi_1(q) = \psi_2(q) = \psi(q)$ (say) for both  $\hat{n} = 0$  and  $\hat{n} = 1$ . Thus the NOT gate is realisable with deformed qubits. Moreover, the conditions for realisation is the same (i.e.  $\psi_1(q) = \psi_2(q) = \psi(q)$ ) as for the Hadamard and Phase Shift gates.

#### 4. The CNOT gate

The Controlled-NOT gate is a two-qubit operator where the first qubit is the control and the second qubit the target. The action of the CNOT gate is defined by the following transformations:

$$|00\rangle \rightarrow |00\rangle ; |01\rangle \rightarrow |01\rangle$$

$$|10\rangle \rightarrow |11\rangle ; |11\rangle \rightarrow |10\rangle$$

where  $|00\rangle \equiv |0\rangle|0\rangle$ ;  $|01\rangle \equiv |0\rangle|1\rangle$  etc. The first line of the transformation signifies that when the control qubit is in the "0"-state, the target qubit does not change after the action of the CNOT gate. The second line means that if the control qubit is in the "1"-state, target qubit changes value after the action of the CNOT gate. This may be written as (modulo constants) as  $|xy\rangle \rightarrow (1-x)|xy\rangle + x|x \quad 1-y\rangle$  i.e.  $|x\rangle|y\rangle \rightarrow (1-x)|x\rangle|y\rangle + x|x\rangle|1-y\rangle$

Let the oscillators corresponding to the  $|x\rangle$  qubit be denoted by  $a, a^\dagger$  and those corresponding to the  $|y\rangle$  qubit be  $b, b^\dagger$ . Then in terms of oscillator states the CNOT transformation reads:

$$\begin{aligned} & \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_a \frac{(b_1^\dagger)^{k_1}(b_2^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}}|\phi\rangle_b \\ & \rightarrow (1-n_1) \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_a \frac{(b_1^\dagger)^{k_1}(b_2^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}}|\phi\rangle_b \\ & + n_1 \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_a \frac{(b_1^\dagger)^{k_2}(b_2^\dagger)^{k_1}}{([k_2]![k_1]!)^{1/2}}|\phi\rangle_b \end{aligned} \quad (16a)$$

where  $n_1, n_2$  and  $k_1, k_2$  are the eigenvalues of the number operators corresponding to the respective oscillators with  $n_1 + n_2 = 1, k_1 + k_2 = 1$  and  $|\phi\rangle_a, |\phi\rangle_b$  denote the ground states corresponding to oscillators  $a_{1,2}$  and  $b_{1,2}$

respectively. Writing,

$$\frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_a = |\eta_1\rangle; \frac{(b_1^\dagger)^{k_1}(b_2^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}}|\phi\rangle_b = |\eta_2\rangle; \frac{(b_1^\dagger)^{k_2}(b_2^\dagger)^{k_1}}{([k_2]![k_1]!)^{1/2}}|\phi\rangle_b = |-\eta_2\rangle$$

the equation (16a) for the CNOT transformation looks like:

$$|\eta_1\rangle|\eta_2\rangle \rightarrow (1 - n_1)|\eta_1\rangle|\eta_2\rangle + n_1|\eta_1\rangle|-\eta_2\rangle \quad (16b)$$

In all subsequent discussions we shall use this form (16b). However, for completeness, we note that the CNOT transformation (16a) can also be written in the alternative notation as

$$|n_1 - \frac{1}{2}\rangle_a |k_1 - \frac{1}{2}\rangle_b \rightarrow (1 - n_1)|n_1 - \frac{1}{2}\rangle_a |k_1 - \frac{1}{2}\rangle_b + n_1|n_1 - \frac{1}{2}\rangle_a |\frac{1}{2} - k_1\rangle_b \quad (16b)$$

i.e.

$$\begin{aligned} |-\frac{1}{2}\rangle_a |-\frac{1}{2}\rangle_b &\rightarrow |-\frac{1}{2}\rangle_a |-\frac{1}{2}\rangle_b ; \quad |-\frac{1}{2}\rangle_a |\frac{1}{2}\rangle_b \rightarrow |-\frac{1}{2}\rangle_a |\frac{1}{2}\rangle_b \\ |\frac{1}{2}\rangle_a |-\frac{1}{2}\rangle_b &\rightarrow |\frac{1}{2}\rangle_a |\frac{1}{2}\rangle_b ; \quad |\frac{1}{2}\rangle_a |\frac{1}{2}\rangle_b \rightarrow |\frac{1}{2}\rangle_a |-\frac{1}{2}\rangle_b \end{aligned}$$

For deformed qubits the CNOT transformation will be

$$|x\rangle_q |y\rangle_q \rightarrow (1 - x)|x\rangle_q |y\rangle_q + x|x\rangle_q |1 - y\rangle_q \quad (17a)$$

or in terms of deformed oscillators :

$$\begin{aligned} &\frac{(a_{1q}^\dagger)^{n_1}(a_{2q}^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_{aq} \frac{(b_{1q}^\dagger)^{k_1}(b_{2q}^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}}|\phi\rangle_{bq} \\ &\rightarrow (1 - n_1) \frac{(a_{1q}^\dagger)^{n_1}(a_{2q}^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_{aq} \frac{(b_{1q}^\dagger)^{k_1}(b_{2q}^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}}|\phi\rangle_{bq} \\ &+ n_1 \frac{(a_{1q}^\dagger)^{n_1}(a_{2q}^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}}|\phi\rangle_{aq} \frac{(b_{1q}^\dagger)^{k_2}(b_{2q}^\dagger)^{k_1}}{([k_2]![k_1]!)^{1/2}}|\phi\rangle_{bq} \end{aligned} \quad (17b)$$

As in Ref.1, the harmonic oscillator realisations for the operators  $a_q, a_q^\dagger$  and  $b_q, b_q^\dagger$  respectively are written in terms of the two functions  $F$  and  $G$  as<sup>11,12</sup>:

$$a_{1q}^\dagger = F(\hat{N}, q) a_1^\dagger \quad ; \quad a_{2q}^\dagger = F(1 - \hat{N}, q) a_2^\dagger \quad (18a)$$

$$b_{1q}^\dagger = G(\hat{K}, q) b_1^\dagger \quad ; \quad b_{2q}^\dagger = G(1 - \hat{K}, q) b_2^\dagger \quad (18b)$$

$\hat{N}$  and  $\hat{K}$  are the respective number operators with eigenvalues  $\hat{n}$  and  $\hat{k}$  and

$$F(\hat{N}, q) = \sqrt{\frac{q^{\hat{N}} \psi_1 - q^{-\hat{N}} \psi_2}{\hat{N}(q - q^{-1})}} \quad , \quad G(\hat{K}, q) = \sqrt{\frac{q^{\hat{K}} \beta_1 - q^{-\hat{K}} \beta_2}{\hat{K}(q - q^{-1})}} \quad , \quad (19)$$

Using these expressions in (17b) (and relabeling  $n_1$  as  $n$  and  $k_1$  as  $k$  etc.) and suppressing the  $q$  dependence in  $F$  and  $G$  to avoid cumbersome notation one gets

$$\begin{aligned} & F^n(\hat{N}) F(1 - \hat{N} + n)^{1-n} \frac{(a_1^\dagger)^n (a_2^\dagger)^{1-n}}{([n]![1-n]!)^{\frac{1}{2}}} |\phi\rangle_{aq} G^k(\hat{K}) G(1 - \hat{K} + k)^{1-k} \frac{(b_1^\dagger)^k (b_2^\dagger)^{1-k}}{([k]![1-k]!)^{\frac{1}{2}}} |\phi\rangle_{bq} \\ & \longrightarrow \\ & (1-n) F^n(\hat{N}, q) F(1 - \hat{N} + n)^{1-n} \frac{(a_1^\dagger)^n (a_2^\dagger)^{1-n}}{([n]![1-n]!)^{\frac{1}{2}}} |\phi\rangle_{aq} G^k(\hat{K}, q) G(1 - \hat{K} + k)^{1-k} \frac{(b_1^\dagger)^k (b_2^\dagger)^{1-k}}{([k]![1-k]!)^{\frac{1}{2}}} |\phi\rangle_{bq} \\ & + n F^n(\hat{N}, q) F(1 - \hat{N} + n)^{1-n} \frac{(a_1^\dagger)^n (a_2^\dagger)^{1-n}}{([n]![1-n]!)^{\frac{1}{2}}} |\phi\rangle_{aq} G^{1-k}(\hat{K}, q) G(1 - \hat{K} + k)^k \frac{(b_1^\dagger)^{1-k} (b_2^\dagger)^k}{([1-k]![k]!)^{\frac{1}{2}}} |\phi\rangle_{bq} \\ & \quad (20a) \end{aligned}$$

Denoting

$$\frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{([n_1]![n_2]!)^{1/2}} |\phi\rangle_{aq} = |\beta_{1q}\rangle; \quad \frac{(b_1^\dagger)^{k_1} (b_2^\dagger)^{k_2}}{([k_1]![k_2]!)^{1/2}} |\phi\rangle_{bq} = |\beta_{2q}\rangle; \quad \frac{(b_1^\dagger)^{k_2} (b_2^\dagger)^{k_1}}{([k_2]![k_1]!)^{1/2}} |\phi\rangle_{bq} = |-\beta_{2q}\rangle$$

$$F^n(\hat{N}) F(1 - \hat{N} + n)^{1-n} = A$$

$$G^k(\hat{K}) G(1 - \hat{K} + k)^{1-k} = B$$

$$G^{1-k}(\hat{K}) G(1 - \hat{K} + k)^k = B'$$

the equation (20a) becomes

$$A|\beta_{1q}\rangle B|\beta_{2q}\rangle \longrightarrow (1-n)A|\beta_{1q}\rangle B|\beta_{2q}\rangle + nA|\beta_{1q}\rangle B'|\beta_{2q}\rangle \quad (20b)$$

Multiplying both sides of (20b) by  $(AB)^{-1}$  gives

$$|\beta_{1q}\rangle |\beta_{2q}\rangle \longrightarrow (1-n)|\beta_{1q}\rangle |\beta_{2q}\rangle + n|\beta_{1q}\rangle B^{-1}B'|\beta_{2q}\rangle \quad (20c)$$

Note that with respect to the states, (20c) will be indistinguishable from the usual CNOT transformation (16b) if  $B^{-1}B' = I$  (the identity operator) i.e.  $B = B'$  or

$$\begin{aligned} & \left( \frac{q^{\hat{K}}\beta_1 - q^{-\hat{K}}\beta_2}{\hat{K}(q - q^{-1})} \right)^{\frac{k}{2}} \left( \frac{q^{1-\hat{K}+k}\beta_1 - q^{-(1-\hat{K}+k)}\beta_2}{(1 - \hat{K} + k)(q - q^{-1})} \right)^{\frac{1-k}{2}} \\ &= \left( \frac{q^{1-\hat{K}}\beta_1 - q^{-(1-\hat{K})}\beta_2}{(1 - \hat{K})(q - q^{-1})} \right)^{\frac{1-k}{2}} \left( \frac{q^{\hat{K}-1+k}\beta_1 - q^{-(\hat{K}-1+k)}\beta_2}{(\hat{K} - 1 + k)(q - q^{-1})} \right)^{\frac{k}{2}} \end{aligned} \quad (21)$$

Equation (21) is an identity for both  $k = 0$  and  $k = 1$ . Therefore the condition  $B = B'$  is always realisable in the domain of  $k$ . So the two qubit CNOT gate can be realised with q-deformed oscillators. Hence all the gates required for universality can also be realised with q-deformed oscillators. This implies that any quantum logic gate can be realised with q-deformed oscillators. Thus quantum computation has an alternative formalism.

## 5. The possible states

There are two possibilities as regards the arbitrary functions  $\psi_{1,2}, \beta_{1,2}$ .

**Case:1** All of them are unity and hence  $N = \hat{N}$  and similarly  $K = \hat{K}$ . So (2a) just relates the operators  $a, a^\dagger$  with  $a_q, a_q^\dagger$ . A similar argument holds for the operators  $b, b^\dagger$  and  $b_q, b_q^\dagger$ . Also from (2b) we then have  $N = \hat{N}$  and  $K = \hat{K}$ . This means that at the occupation number level the deformed states

cannot be distinguished from the undeformed states and we are in the realm of standard quantum computation. We denote eigenvalues of the number operators for deformed oscillators in Case I by  $n, k$  ( $\hat{n}, \hat{k}$  still correspond to undeformed oscillators); the states in Case I by  $|\rangle_I$ . Then relabel  $n_1$  by  $n$  etc. we have for Case I

$$\begin{aligned} |n - 1/2\rangle_I |k - 1/2\rangle_I &= |n\rangle_{Ia_1} |1 - n\rangle_{Ia_2} |k\rangle_{Ib_1} |1 - k\rangle_{Ib_2} \\ &= |\hat{n}\rangle_{Ia_1} |1 - \hat{n}\rangle_{Ia_2} |\hat{k}\rangle_{Ib_1} |1 - \hat{k}\rangle_{Ib_2} \end{aligned} \quad (22a)$$

where  $n = 0, 1; k = 0, 1$  and  $n = \hat{n}; k = \hat{k}$ . Note that all states have  $j = \frac{1}{2}$ . The possible states are:

$$|00\rangle_I = |-\frac{1}{2} \quad -\frac{1}{2}\rangle_I = |-\frac{1}{2}\rangle_{Ia} |-\frac{1}{2}\rangle_{Ib} = |\tilde{0}\rangle_{Ia_1} |\tilde{1}\rangle_{Ia_2} |\tilde{0}\rangle_{Ib_1} |\tilde{1}\rangle_{Ib_2} \quad (22b)$$

$$|01\rangle_I = |-\frac{1}{2} \quad \frac{1}{2}\rangle_I = |-\frac{1}{2}\rangle_{Ia} |\frac{1}{2}\rangle_{Ib} = |\tilde{0}\rangle_{Ia_1} |\tilde{1}\rangle_{Ia_2} |\tilde{1}\rangle_{Ib_1} |\tilde{0}\rangle_{Ib_2} \quad (22c)$$

$$|10\rangle_I = |\frac{1}{2} \quad -\frac{1}{2}\rangle_I = |\frac{1}{2}\rangle_{Ia} |-\frac{1}{2}\rangle_{Ib} = |\tilde{1}\rangle_{Ia_1} |\tilde{0}\rangle_{Ia_2} |\tilde{0}\rangle_{Ib_1} |\tilde{1}\rangle_{Ib_2} \quad (22d)$$

$$|11\rangle_I = |\frac{1}{2} \quad \frac{1}{2}\rangle_I = |\frac{1}{2}\rangle_{Ia} |\frac{1}{2}\rangle_{Ib} = |\tilde{1}\rangle_{Ia_1} |\tilde{0}\rangle_{Ia_2} |\tilde{1}\rangle_{Ib_1} |\tilde{0}\rangle_{Ib_2} \quad (22e)$$

## Case:II

We have a general scenario if the arbitrary functions  $\psi_i(q), \beta_i(q) i = 1, 2$  are not all equal to unity. As these are arbitrary, let us choose  $\psi_1 = \psi_2 = \psi, \beta_1 = \beta_2 = \beta$ . Then  $N = \hat{N} - (1/s) \ln \psi(q)$  [(2b)]; and  $K = \hat{K} - (1/s) \ln \beta(q)$ .

Hence states labelled by the occupation number are different as the eigenvalues of the number operator of standard oscillator states and the eigenvalues of the number operator of deformed oscillator states are now related by

$n = \hat{n} - (1/s) \ln \psi(q)$  ;  $k = \hat{k} - (1/s) \ln \beta(q)$ . This would show up in the Jordan-Schwinger construction. We denote eigenvalues of the number operators for deformed oscillators in Case II by  $n', k'$  and the states by  $| >_{II}$ . So

$$\begin{aligned}
& |n' - 1/2\rangle_{II} |k' - 1/2\rangle_{II} \\
& = |n'\rangle_{IIa_1} |1 - n'\rangle_{IIa_2} |k'\rangle_{IIb_1} |1 - k'\rangle_{IIb_2} \\
& = |\hat{n} - (1/s) \ln \psi\rangle_{IIa_1} |1 - \hat{n} + (1/s) \ln \psi\rangle_{IIa_2} |\hat{k} - (1/s) \ln \beta\rangle_{IIb_1} |1 - \hat{k} + (1/s) \ln \beta\rangle_{IIb_2}
\end{aligned} \tag{23a}$$

All possible states are :

$$|00\rangle_{II} = |-\frac{1}{2} \quad -\frac{1}{2}\rangle_{II} = |-\frac{1}{2}\rangle_{IIa} |-\frac{1}{2}\rangle_{IIb} = |\tilde{0}\rangle_{IIa_1} |\tilde{1}\rangle_{IIa_2} |\tilde{0}\rangle_{IIb_1} |\tilde{1}\rangle_{IIb_2} \tag{23b}$$

$$|01\rangle_{II} = |-\frac{1}{2} \quad \frac{1}{2}\rangle_{II} = |-\frac{1}{2}\rangle_{IIa} |\frac{1}{2}\rangle_{IIb} = |\tilde{0}\rangle_{IIa_1} |\tilde{1}\rangle_{IIa_2} |\tilde{1}\rangle_{IIb_1} |\tilde{0}\rangle_{IIb_2} \tag{23c}$$

$$|10\rangle_{II} = |\frac{1}{2} \quad -\frac{1}{2}\rangle_{II} = |\frac{1}{2}\rangle_{IIa} |-\frac{1}{2}\rangle_{IIb} = |\tilde{1}\rangle_{IIa_1} |\tilde{0}\rangle_{IIa_2} |\tilde{0}\rangle_{IIb_1} |\tilde{1}\rangle_{IIb_2} \tag{23d}$$

$$|11\rangle_{II} = |\frac{1}{2} \quad \frac{1}{2}\rangle_{II} = |\frac{1}{2}\rangle_{IIa} |\frac{1}{2}\rangle_{IIb} = |\tilde{1}\rangle_{IIa_1} |\tilde{0}\rangle_{IIa_2} |\tilde{1}\rangle_{IIb_1} |\tilde{0}\rangle_{IIb_2} \tag{23e}$$

Consistency demands the following interpretations:

(1) For (23b),  $\psi = q^{\hat{n}}$ ;  $\beta = q^{\hat{k}}$  i.e. the qubit state  $|\tilde{0}\rangle_{IIa_1} |\tilde{1}\rangle_{IIa_2}$  corresponds to an oscillator occupation number  $\hat{n} > 0$  while  $|\tilde{0}\rangle_{IIb_1} |\tilde{1}\rangle_{IIb_2}$  corresponds to an oscillator occupation number  $\hat{k} > 0$ .

(2) For (23c),  $\psi = q^{\hat{n}}$ ;  $\beta = q^{\hat{k}-1}$  i.e. the qubit state  $|\tilde{0}\rangle_{IIa_1} |\tilde{1}\rangle_{IIa_2}$  corresponds to an oscillator occupation number  $\hat{n} > 0$  while  $|\tilde{1}\rangle_{IIb_1} |\tilde{0}\rangle_{IIb_2}$  corresponds to an oscillator occupation number  $\hat{k} > 1$ .

(3) For (23d),  $\psi = q^{\hat{n}-1}$ ;  $\beta = q^{\hat{k}}$  i.e. the qubit state  $|\tilde{1}\rangle_{IIa_1}|\tilde{0}\rangle_{IIa_2}$  corresponds to an oscillator occupation number  $\hat{n} > 1$  while  $|\tilde{0}\rangle_{IIb_1}|\tilde{1}\rangle_{IIb_2}$  corresponds to an oscillator occupation number  $\hat{k} > 0$ .

(4) For (23e),  $\psi = q^{\hat{n}-1}$ ;  $\beta = q^{\hat{k}-1}$  i.e. the qubit state  $|\tilde{1}\rangle_{IIa_1}|\tilde{0}\rangle_{IIa_2}$  corresponds to an oscillator occupation number  $\hat{n} > 1$  while  $|\tilde{1}\rangle_{IIb_1}|\tilde{0}\rangle_{IIb_2}$  corresponds to an oscillator occupation number  $\hat{k} > 1$ .

Therefore we always have  $\hat{n} > n', \hat{k} > k'$ .  $\psi(q), \beta(q)$  cannot be unity (i.e.  $\hat{n}, \hat{k}$  cannot be zero) because then we will have  $n' = \hat{n}, k' = \hat{k}$  i.e. Case I. So the deformed states in Case II can be related to harmonic oscillator states with occupation numbers greater than zero.

Denote the  $F$  and  $G$  functions corresponding to the two possibilities by  $F_I, G_I$  and  $F_{II}, G_{II}$ . Then

$$F_I(\hat{N}, q) = \left( \frac{q^{\hat{N}} - q^{-\hat{N}}}{\hat{N}(q - q^{-1})} \right)^{\frac{1}{2}} ; \quad G_I(\hat{K}, q) = \left( \frac{q^{\hat{K}} - q^{-\hat{K}}}{\hat{K}(q - q^{-1})} \right)^{\frac{1}{2}} \quad (24)$$

$$F_{II}(\hat{N}, q) = \left( \frac{q^{\hat{N}}\psi_1 - q^{-\hat{N}}\psi_2}{\hat{N}(q - q^{-1})} \right)^{\frac{1}{2}} ; \quad G_{II}(\hat{K}, q) = \left( \frac{q^{\hat{K}}\beta_1 - q^{-\hat{K}}\beta_2}{\hat{K}(q - q^{-1})} \right)^{\frac{1}{2}} \quad (25)$$

where we have labelled the arbitrary functions by  $\psi_{1,2}$  and  $\beta_{1,2}$ . Now the properties of the operators  $F$  and  $G$  have to be understood in terms of their eigenvalues. Then the ratio of the eigenvalues of  $F_{II}$  and  $F_I$  is (choosing  $\psi_1 = \psi_2 = \psi$  and  $\beta_1 = \beta_2 = \beta$ )

$$\frac{\text{Eigenvalue of } F_{II}}{\text{Eigenvalue of } F_I} = \left( \frac{q^{2\hat{n}}\psi_1(q) - \psi_2(q)}{q^{2\hat{n}} - 1} \right)^{1/2} = \psi^{\frac{1}{2}}(q) \quad (26a)$$

So we may write

$$F_{II} = \psi^{\frac{1}{2}}(q) F_I \quad (26b)$$



Similarly

$$\frac{\text{Eigenvalue of } G_{II}}{\text{Eigenvalue of } G_I} = \left( \frac{q^{2\hat{k}}\beta_1(q) - \beta_2(q)}{q^{2\hat{k}} - 1} \right)^{1/2} = \beta^{\frac{1}{2}}(q) \quad (27a)$$

So we may write

$$G_{II} = \beta^{\frac{1}{2}}(q)G_I \quad (27b)$$

Thus

$$\begin{aligned} |n', k'\rangle_{II} &= \frac{(F_{II}a_1^\dagger)^{n'_1}(F_{II}a_2^\dagger)^{n'_2}}{([n'_1]![n'_2]!)^{1/2}}|\phi\rangle_{aq} \frac{(G_{II}b_1^\dagger)^{k'_1}(G_{II}b_2^\dagger)^{k'_2}}{([k'_1]![k'_2]!)^{1/2}}|\phi\rangle_{bq} \\ &= \frac{(\psi^{\frac{1}{2}}F_I(\hat{N})a_1^\dagger)^{n'}(\psi^{\frac{1}{2}}F_I(1-\hat{N})a_2^\dagger)^{1-n'}}{([n']![1-n']!)^{1/2}}|\phi\rangle_{aq} \\ &\quad \frac{(\beta^{\frac{1}{2}}G_I(\hat{K})b_1^\dagger)^{k'}(\beta^{\frac{1}{2}}G_I(1-\hat{K})b_2^\dagger)^{1-k'}}{([k']![1-k']!)^{1/2}}|\phi\rangle_{bq} \\ &= \psi^{\frac{n'}{2}}\psi^{\frac{1-n'}{2}}\beta^{\frac{k'}{2}}\beta^{\frac{1-k'}{2}}|n, k\rangle_I = \psi^{\frac{1}{2}}\beta^{\frac{1}{2}}|n, k\rangle_I \end{aligned} \quad (28)$$

Therefore

$$\begin{aligned} &\frac{{}_{II}\langle n', k'|n', k'\rangle_{II}}{{}_I\langle n, k|n, k\rangle_I} \\ &= \psi^{\frac{1}{2}}(q)\beta^{\frac{1}{2}}(q) \end{aligned} \quad (29)$$

So the right hand side of (29) is a function of  $q$  only. For  $\psi(q) = 1$  and  $\beta(q) = 1$ , one cannot distinguish between the two cases at the level of the scalar products between states. However, if the arbitrary functions are not unity then these scalar products are distinct from each other and this might be useful at the level of experimental realisations or consequences. Moreover, note that one can choose the arbitrary function  $\psi(q)$  for the control qubit to be the same as that in the Hadamard, Phase shift and NOT gates. As the CNOT gate is a two qubit gate, a different function  $\beta(q)$  is taken for the

target qubit. So with two arbitrary functions all the three gates required for universality can be constructed with  $q$ -deformed qubits

## 6. Conclusion

Therefore quantum computation admits of an alternative formalism where  $q$ -deformed oscillator states can be used to construct qubits. This has been established here with the realisation of the CNOT quantum logic gate with  $q$ -deformed oscillators. Thus this realisation is possible for all the quantum logic gates required for universality. Hence *all quantum logic gates can be realised with  $q$ -deformed qubits*. The existence of additional parameters will enable comparison between different experimental scenarios using the usual scheme and the alternate scheme. This requires further study.

Another aspect where the new formalism might prove useful is the realm of quantum error correction. The conventional way to correct computer error is to use redundancy. More than one element is used to denote the same bit. For example, consider two atoms  $A$  and  $B$  and use the doublet  $AB$  to store the same bit (of information). Thus one has the state  $|00\rangle = |0\rangle|0\rangle$  (or the state  $|11\rangle = |1\rangle|1\rangle$ ). An error changes the state of only one atom, i.e. one now has any one of the states  $|01\rangle$ ,  $|10\rangle$  instead of  $|00\rangle$  (or the states  $|10\rangle$ ,  $|01\rangle$  instead of  $|11\rangle$ ). For type II states,  $|n\rangle_{II}$ , we have  $n' = \hat{n} - \frac{1}{s} \ln \psi$ . It is simple to check that  $\psi = q^{\hat{n}}$  for  $n' = 0$  and  $\psi = q^{\hat{n}-1}$  for  $n' = 1$ . Therefore occurrence of an error is reflected in the change of the arbitrary function and consequently in the matrix elements  ${}_{II}\langle n'|n'\rangle_{II}$ . So the arbitrary functions provide an additional leverage and can possibly be used in detecting and regulating errors. This also requires further investigation.

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